

Answer All Questions

Q1. [8 points] If $n^2 + 2$ is odd, then $n + 1$ is not odd.

$\forall n \in \mathbb{Z}$ $n^2 + 2$ is odd $\rightarrow n + 1$ is even
contrapositive

$\forall n \in \mathbb{Z}$ $n + 1$ is odd $\rightarrow n^2 + 2$ is even

Let n is integer and $n + 1$ is odd

$$n + 1 = 2k + 1$$

$$n = 2k$$

$$n^2 = 4k^2$$

$$n^2 = 4k^2$$

$$n + 1 = 2k + 1$$

$$n = 2k$$

$$n = 2k$$

$$n^2 = 4k^2$$

$$n^2 + 2 = 4k^2 + 2$$

$$n^2 + 2 = 2(2k^2 + 1)$$

$$n^2 + 2 = 2m \quad \text{where } m = 2k^2 + 1 \in \mathbb{Z}$$

$\therefore n^2 + 2$ is even Hence proved.

Q2. (a) [4 points] Determine all of the elements in each of the following sets.

$$(1) \{1 + (-1)^n \mid n \in \mathbb{Z}\} = \{2, 0\}$$

$$(2) \{x \in \mathbb{R} \mid x^2 + 4x + 3 = 0\} = \{-1, -3\} \quad x = \frac{-4 \pm \sqrt{16 - 12}}{2} \Rightarrow \begin{cases} x = -1 \\ -3 = x \end{cases}$$

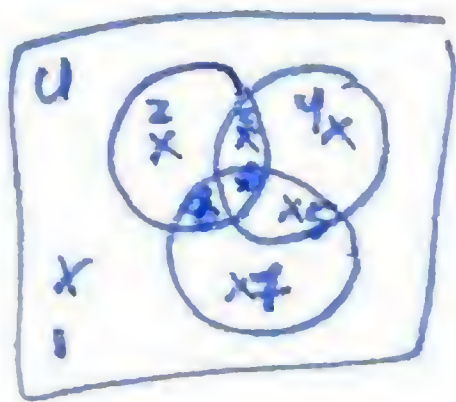
(b) [6 points] Let $A_n = \{n + 1, n + 2, n + 3, \dots\}$. Find $A_{n+1} = \{n + 2, n + 3, n + 4, \dots\}$

$$(1) A_n \cup A_{n+1} = \{n + 1, n + 2, n + 3, \dots\} = A_n$$

$$(2) A_n \cap A_{n+1} = \{n + 2, n + 3, n + 4, \dots\} = A_{n+1}$$

$$(3) |A_n \cap P(A_n)| = A_n \cap P(A_n) = \emptyset \quad P(A_n) = \{\emptyset, \{n + 1\}, \dots\}$$

$$\therefore |A_n \cap P(A_n)| = 0 \text{ zero.}$$



$$A \cap B = \{3, 6\}$$

$$A \cap B - C = \{3\}$$

$$A \cap C = \{6, 8\}$$

$$A \cap C - B = \{8\}$$

disjoints

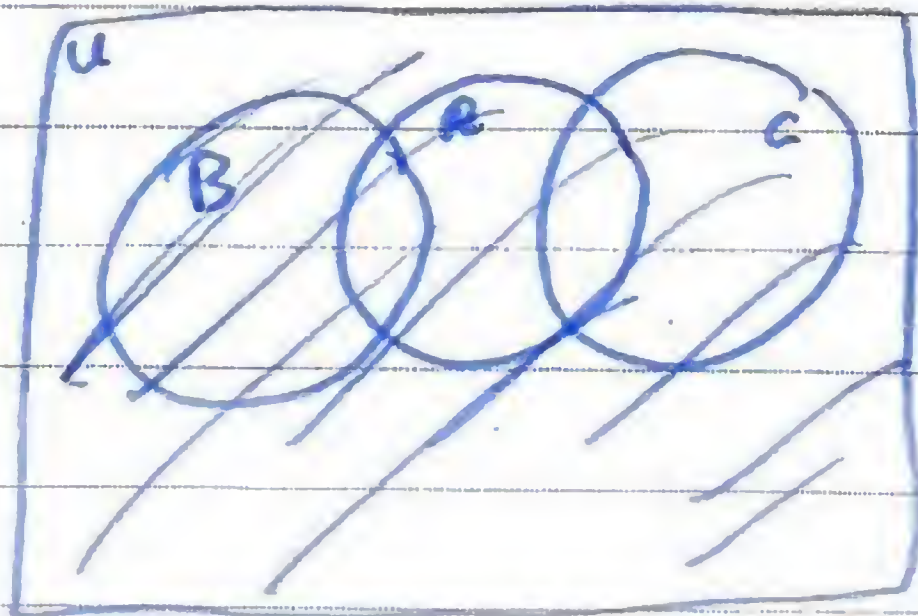
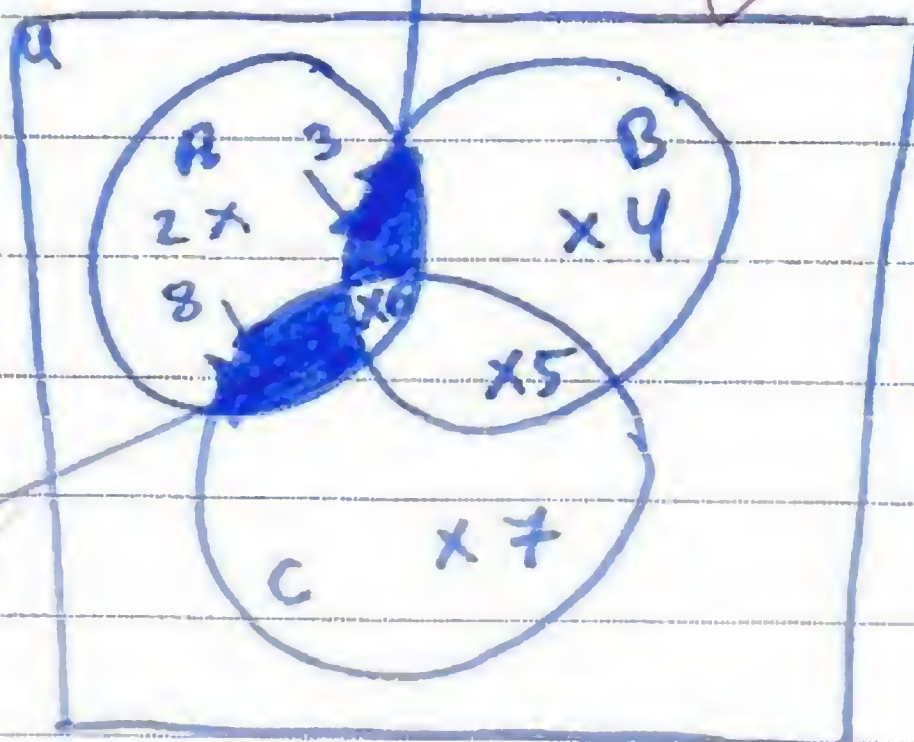
$$(A \cap B) - C \neq (A \cap C) - B$$

Q3. [8 points] For any sets A , B , and C , show that the sets $(A \cap B) - C$ and $(A \cap C) - B$ are disjoint.

same as up.

$$A \cap B - C \Rightarrow \{3\}$$

$$A \cap C - B = \{8\}$$



6 $\therefore (A \cap B) - C$ and $(A \cap C) - B$ are disjoint How?

Q4. [9 points] Let $a, b, c \in \mathbb{Z}$ and $c \neq 0$. Prove that if c divides $(ca + b)$, then c divides b .

$$\forall a, b, c \in \mathbb{Z} \quad c \text{ divides } (ca + b) \rightarrow c \text{ divides } b$$

To conclude

Assume: c divides $(ca + b)$

$\therefore ca + b$ is divisible by c

Let c is an integer and $c \neq 0$ and

Let $ca + b$ an integer and divisible by c

$$\therefore ca + b = ck \quad k \in \mathbb{Z}$$

$$b = ck - ca$$

$$b = c(k - a)$$

$$b = cm \quad \text{where } m \in \mathbb{Z} \text{ and } m = (k - a)$$

$\therefore b$ is divisible by c

$\therefore c$ divides b

Hence proved.

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Q5. [10 points] Prove that

$$\frac{2}{3} + \frac{2}{3^2} + \dots + \frac{2}{3^n} = 1 - \left(\frac{1}{3}\right)^n, \quad n \geq 1$$

Let $P(n) : \frac{2}{3} + \frac{2}{3^2} + \dots + \frac{2}{3^n} = 1 - \left(\frac{1}{3}\right)^n \quad n \geq 1$

Basis: Show $P(1) : \frac{2}{3} = 1 - \left(\frac{1}{3}\right)^1 \Rightarrow \frac{2}{3} = \frac{2}{3}$

$\therefore R.H.S. = L.H.S \quad \therefore P(1) \text{ is true}$

Assumption $P(k) : \frac{2}{3} + \frac{2}{3^2} + \dots + \frac{2}{3^k} = 1 - \left(\frac{1}{3}\right)^k \quad k \geq 1$

Induction Show $P(k+1) : \frac{2}{3} + \frac{2}{3^2} + \dots + \frac{2}{3^k} + \frac{2}{3^{k+1}} = 1 - \left(\frac{1}{3}\right)^{k+1} \quad k \geq 1$

L.H.S $\Rightarrow \left\{ \frac{2}{3} + \frac{2}{3^2} + \dots + \frac{2}{3^k} \right\} + \frac{2}{3^{k+1}} \quad k \geq 1$

from assumption $\Downarrow 1 - \left(\frac{1}{3}\right)^k + \frac{2}{3^{k+1}} \quad k \geq 1$

$1 - \left(\frac{1}{3}\right)^k + \frac{2}{3^k \cdot 3} \Rightarrow 1 - \frac{1^k}{3^k} + \frac{2}{3^k \cdot 3} \quad k \geq 1$

$1 - \frac{1^k \cdot 3 + 2}{3^k \cdot 3} \Rightarrow$

$\frac{1 \cdot 3^k - 1^k}{3^k} + \frac{2}{3^k \cdot 3} \quad k \geq 1$

$\Rightarrow \frac{(3^k - 1)3}{3^k \cdot 3} + \frac{2}{3^k \cdot 3} \quad k \geq 1$

$\frac{3^{k+1} - 3 + 2}{3^{k+1}} \Rightarrow \frac{3^{k+1} - 1}{3^{k+1}} \Rightarrow 1 - \left(\frac{1}{3}\right)^{k+1} \quad k \geq 1$

$\therefore P(k+1)$
hence proved.